CS/MATH 3414 Homework # 2

A Summation Problem or
Why a Little Mathematics is Better than a Lot of Dumb Computing

For $0 \leq x \leq 1$, consider the function defined by

$$
\phi(x) = \sum_{k=1}^{\infty} \frac{1}{k(k + x)}.
$$

Try to evaluate $\phi(0)$ (say to 8 significant digits) directly by computing partial sums of the series. This can be done with Mathematica by

$$
p[x_, \text{ limit_}] := \text{NSum}[ 1/(k(k + x))], \{k, \text{ limit, 1, -1} \}, \text{NSumTerms} \to \text{ limit }]
$$

and then, for example,

$$
p[0, 20]
$$
gives the partial sum with 20 terms approximating $\phi(0)$.

Suggestion: figure out what $n$ should be such that

$$
\sum_{k=n+1}^{\infty} \frac{1}{k^2} < 10^{-8}.
$$

Then compute $p[0, n]$ Do you think this partial sum will in fact be accurate to $10^{-8}$?
A Summation Problem (Part 2)

Consider now a series derived from $\phi(x)$ that converges much faster than the series for $\phi(x)$. Let

$$\psi(x) = \frac{\phi(x) - \phi(1)}{1 - x} = \sum_{k=1}^{\infty} \frac{1}{k(k+1)(k+x)}.$$  

Note that $\psi(2)$ can be evaluated exactly by figuring out the partial fraction expansion

$$\frac{1}{k(k+1)(k+2)} = \frac{A}{k} + \frac{B}{k+1} + \frac{C}{k+2}.$$  

Then the series

$$\frac{\psi(x) - \psi(2)}{2 - x} = \sum_{k=1}^{\infty} \frac{1}{k(k+1)(k+2)(k+x)},$$

which converges rapidly, can be evaluated numerically. Finally, since $\phi(1)$ and $\psi(2)$ are known, $\phi(x)$ can be deduced.

To determine how many terms to use in the sum, note that the error in the partial sum

$$\sum_{k=1}^{N} \frac{1}{k(k+1)(k+2)(k+x)},$$

which is what you compute, is bounded by

$$\sum_{k=N+1}^{\infty} \frac{1}{k(k+1)(k+2)(k+x)} \leq \sum_{k=N+1}^{\infty} \frac{1}{k^4} \leq \int_{N+1}^{\infty} \frac{dx}{x^4} = \frac{1}{3N^3} \leq .5 \cdot 10^{-7}.$$  

Solving the last inequality for $N$ tells you how many terms you need to meet the accuracy requirement.

For your information, $\phi(0) = 1.64493 \ 40668 \ 48226.$

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