CS/MATH 3414 Homework # 10

(6) 1. Compute \( w_i \) and \( x_i \) \((i = 0, 1, 2)\) such that the formula

\[
\int_{-1}^{1} f(x) \cos \left( \frac{\pi x}{2} \right) \, dx \approx \sum_{i=0}^{2} w_i f(x_i)
\]

is exact if \( f \) is a polynomial of degree \( \leq 5 \). Give \( w_i \) and \( x_i \) to 12 significant digits.

(4) 2. Compute coefficients such that the formula

\[
\int_{0}^{\infty} f(x) e^{-x} \, dx \approx w_0 f(0) + \tilde{w}_0 f'(0) + w_2 f(1) + \tilde{w}_2 f'(1)
\]

is exact if \( f \) is a polynomial of degree \( \leq 3 \). [Hint: \( \int_{0}^{\infty} x^n e^{-x} \, dx = \Gamma(n + 1) = n! \)]

(2) 3. Compute the first three orthogonal polynomials \( \Omega_0, \Omega_1, \Omega_2 \) with respect to the inner product \( \langle f, g \rangle = -\int_{0}^{1} f(x) g(x) \ln x \, dx \).

Extra Credit

(3) 4. Express \( x^5 \) as a linear combination of the Chebyshev polynomials \( T_0, T_1, \ldots, T_5 \):

\[
x^5 = \text{________}_T_0 + \text{________}_T_1 + \text{________}_T_2 + \text{________}_T_3 + \text{________}_T_4 + \text{________}_T_5.
\]

(5) 5. Find the polynomial \( p^*(x) \) of degree \( \leq 3 \) which minimizes

\[
\| p(x) - x^5 \|^2 = \int_{-1}^{1} \frac{(x^5 - p(x))^2}{\sqrt{1 - x^2}} \, dx
\]

over all polynomials \( p(x) \) of degree \( \leq 3 \).
LEAST SQUARES APPROXIMATION EXAMPLE

Problem: Find the polynomial $p^*(x)$ of degree $\leq 2$ which minimizes

$$
\int_{-1}^{1} (p(x) - \sin x)^2 \, dx.
$$

Solution: Define $\langle r, s \rangle = \int_{-1}^{1} r(x)s(x) \, dx$, $f(x) = \sin x$. Then the problem is equivalent to

$$
\min_p \int_{-1}^{1} (p(x) - \sin x)^2 \, dx = \min_p \langle p - f, p - f \rangle = \min_p \|p - f\|^2
$$

(1)

or

$$
\min_p \|p - f\|
$$

(2)

where the minimization is done over all polynomials $p$ of degree $\leq 2$. The solution to (2) is

$$
p^* = \sum_{i=0}^{2} \frac{\langle f, \phi_i \rangle}{\langle \phi_i, \phi_i \rangle} \phi_i
$$

(3)

where $\phi_0, \phi_1, \phi_2$ are orthogonal polynomials with respect to the inner product $\langle \cdot, \cdot \rangle$. Now, working out the details, $\phi_0 = 1$, $\phi_1 = x$, $\phi_2 = (3/2)(x^2 - 1/3)$, $\langle \phi_0, \phi_0 \rangle = 2$, $\langle \phi_1, \phi_1 \rangle = 2/3$, $\langle \phi_2, \phi_2 \rangle = 2/5$, $\langle f, \phi_0 \rangle = 0$, $\langle f, \phi_1 \rangle = 2(\sin 1 - \cos 1)$, $\langle f, \phi_2 \rangle = 0$. Thus

$$
p^*(x) = 0 \cdot 1 + \frac{2(\sin 1 - \cos 1)}{2/3} x + 0 \cdot \frac{3}{2}(x^2 - 1/3) = 3(\sin 1 - \cos 1)x.
$$