CS/MATH 3414 Homework # 1

In 250 B. C. E., the Greek mathematician Archimedes estimated the number \( \pi \) as follows. He looked at a circle with diameter 1, and hence circumference \( \pi \). Inside the circle he inscribed a square. The perimeter of the square is smaller than the circumference of the circle, and so it is a lower bound for \( \pi \). Archimedes then considered an inscribed octagon, 16-gon, etc., each time doubling the number of sides of the inscribed polygons, and producing ever better estimates for \( \pi \). Using 96-sided inscribed and circumscribed polygons, he was able to show that \( 223/71 < \pi < 22/7 \). There is a recursive formula for these estimates. Let \( p_n \) be the perimeter of the inscribed polygon with \( 2^n \) sides. Then \( p_2 \) is the perimeter of the inscribed square, \( p_2 = 2\sqrt{2} \). In general

\[
p_{n+1} = 2^n \sqrt{2(1 - \sqrt{1 - (p_n/2^n)^2})},
\]

\[
p_2 = 2\sqrt{2}.
\]

Compute \( p_n \) for \( n = 3, 4, \ldots, 60 \). Try to explain the failure in the formula. (This problem was suggested by Alan Cline.)

The formula derived above to estimate \( \pi \) fails due to a combination of underflow and catastrophic cancellation. The formula can be improved so that the subtraction is eliminated. First write \( p_{n+1} \) as

\[
p_{n+1} = 2^n \sqrt{r_{n+1}},
\]

where

\[
r_{n+1} = 2(1 - \sqrt{1 - (p_n/2^n)^2}), \quad r_3 = 2/(2 + \sqrt{2}).
\]

Show that

\[
r_{n+1} = \frac{r_n}{2 + \sqrt{4 - r_n}}.
\]

Use the last iteration to calculate \( r_n \) and \( p_n \) for \( n = 3, 4, \ldots, 60 \). (This revision was suggested by W. Kahan.)

Eventually, \( 4 - r_n \) will round to 4, and so the latter formula is still affected by rounding errors for large values of \( n \). Should this concern us?