Inductive Reasoning

Strictly speaking, all our knowledge outside mathematics… consists of conjectures.

…

There are, of course, conjectures and conjectures. There are highly respectable and reliable conjectures as those expressed in certain general laws of physical science.

There are other conjectures, neither reliable nor respectable, some of which may make you angry when you read them in a newspaper.

And in between there are all sorts of conjectures, hunches, and guesses.

George Polya

Inducing a Hypothesis

Inductive reasoning draws generalized conclusions from a finite collection of specific observations.

The act of inducing a conclusion should provide some degree of support for the truth of the conclusion, but cannot ensure it.

Specific observation:
Every student I have encountered at Virginia Tech is a hominid.

Generalized conclusion:
All students at Virginia Tech are hominids.

The conclusion may be strengthened by an indication of how long I have been at Virginia Tech and how many students I have encountered there.

Nevertheless… the conclusion cannot be absolutely affirmed.

Some Kinds of Inductive Reasoning

Simple Induction

reason from observations of a sample group to a conclusion about a member of the larger population

17 of 20 dentists we surveyed majored in Mining Eng as undergrads.

So, there is a high probability, around 85%, that your dentist will have majored in Mining Eng as an undergrad.

This is related to *inductive generalization*, which would conclude from the same evidence that:

So, 85% of all dentists majored in Mining Eng as undergrads.

The Problem of Induction

Does inductive reasoning lead to knowledge?

What justifies either of the following:

Generalizing about the properties of a class of objects based on some number of observations of particular instances of that class.

For example, the inference that "all swans we have seen are white, and therefore all swans are white," before the discovery of black swans.

Presupposing that a sequence of events in the future will occur as it always has in the past.

For example, that the laws of physics will hold as they have always been observed to hold.

The Problem of Induction

The problems just considered have a hallowed history in Western philosophy, from the Greeks to David Hume and beyond…

In the end, perhaps Hume put it best:

> If we be, therefore, engaged by arguments to put trust in past experience, and make it the standard of our future judgement, these arguments must be probable only, or such as regard matter of fact and real existence, according to the division above mentioned. But that there is no argument of this kind, must appear, if our explication of that species of reasoning be admitted as solid and satisfactory.
>
> We have said that all arguments concerning existence are founded on the relation of cause and effect; that our knowledge of that relation is derived entirely from experience; and that all our experimental conclusions proceed upon the supposition that the future will be conformable to the past.
>
> To endeavour, therefore, the proof of this last supposition by probable arguments, or arguments regarding existence, must be evidently going in a circle, and taking that for granted, which is the very point in question.

An Enquiry Concerning Human Understanding

The Necessity of Induction

So, does inductive reasoning lead to knowledge?

In some instances, perhaps… and in other instances, perhaps not.

However, there seems no alternative if we are to form any conclusions regarding real phenomena.

We may observe and induce, and maintain eternal vigilance regarding the correctness of the conclusions we induce.

Sequence Example

A common test question requiring induction is to determine the next value in an incomplete sequence of integers. For example:

11, 13, 17, 19, ??

What would the next value be?

23?

31?

Something else?

The key to induction is to look for patterns… sometimes there is no obvious pattern… and even an obvious pattern is never unique.

Bachet's Conjecture

Doodling with small integers, you might notice that:

1	1
2	1 + 1
3	1 + 1 + 1
4	4
5	4 + 1
6	4 + 1 + 1
7	4 + 1 + 1 + 1
8	4 + 4
9	9
10	9 + 1

So what? Well, so far, every positive integer we've looked at can be written as a sum of perfect squares, and we've never needed more than four perfect squares.

Turns out the conjecture holds up if you continue the table…

Bachet's Conjecture

One more:

325   289 + 16 + 16 + 4

Fine… so we might (if we were Claude Gaspard Bachet de Méziriac) conjecture that:

Any positive integer is a square, or the sum of two, three, or four squares.

And… if you were Joseph Louis Lagrange, you might prove it.

Euler's Formula

Leonhard Euler noticed that if we define

$$p(n) = n^2 - n + 41$$

then the values $p(1)$, $p(2)$, …, $p(20)$ are

41, 43, 47, 53, 61, 71, 83, 97, 113, 131, 151,
173, 197, 223, 251, 281, 313, 347, 383, 421

all of which are prime.

So, you might conjecture…

And… if you consider $p(41)$ you will realize you are wrong…

Final Note

Do not confuse inductive reasoning with mathematical induction.

Despite some superficial similarities, they are fundamentally different.

Logic is the anatomy of thought.

John Locke

When introduced at the wrong time or place, good logic may be the worst enemy of good teaching.

George Polya

Propositions

A *proposition* is a statement that is true or false:

Di-hydrogen monoxide is liquid at a temperature of 0^{o} Fahrenheit.

The Gregorian calendar uses twelve months.

Every even integer greater than 2 can be expressed as the sum of two prime integers.

Not all declarative statements are propositions:

Fried chicken livers taste wonderful.

This sentence is false.

Propositional Logic

Propositional logic is concerned with the "algebra" of propositions.

We may form new propositions from old ones by performing certain "algebraic" operations. Consider the following two rather dull propositions:

51 is a prime integer.

The set {3, 5, 7, 11} contains only prime integers.

Then the following are also propositions:

51 is <u>not</u> a prime integer.

51 is a prime integer <u>or</u> the set {3, 5, 7, 11} contains only prime integers.

51 is a prime integer <u>and</u> the set {3, 5, 7, 11} contains only prime integers.

And so are the following:

If 51 is a prime integer, then the set {3, 5, 7, 11} contains only prime integers.

If the set {3, 5, 7, 11} contains only prime integers, then 51 is a prime integer.

If the set {3, 5, 7, 51} contains only prime integers, then 51 is a prime integer.

Logical Connectives

Suppose that P and Q are propositions, then so are the following:

not P	*negation*	$\neg P$
P and Q	*conjunction*	$P \wedge Q$
P or Q	*disjunction*	$P \vee Q$
if P then Q	*implication*	$P \rightarrow Q$

True or False?

Suppose that P and Q are propositions, then:

not P	is true if and only if P is false
P and Q	is true if and only if P is true and Q is true
P or Q	is true if and only if P is true, or Q is true, or both are true
if P then Q	is true if and only if P is false or Q is true

P	not P
T	F
F	T

P	Q	P and Q
T	T	T
T	F	F
F	T	F
F	F	F

P	Q	P or Q
T	T	T
T	F	T
F	T	T
F	F	F

P	Q	if P then Q
T	T	T
T	F	F
F	T	T
F	F	T

Notes:

- "or" is *inclusive*; that is it's true as long as at least one "side" is true

- "if… then" is false only in the case that the *antecedent* is true and the *consequent* is false

- "if… then" has absolutely nothing to do with causation

- If you're confused by the truth table for "if…then", consider such statements as being a contract. If you say

  If today is Wednesday then I will give you a dollar.

  precisely what are your obligations if today is not Wednesday?

Tautologies

A propositional expression is a *tautology* if the expression is always true, no matter what truth values its components may have.

Example: P or not P.

Now, if P is true then not P must be false, and therefore the expression is true (by definition of "or").

On the other hand, if P is false then not P must be true, and again the expression is true.

This may also be expressed by the construction of a *truth table* that considers all cases:

P	P or not P
T	T
F	T

Inference Rules

An *inference rule* is rule that allows us to infer a conclusion from given conditions (the *premise*).

Inference rules take the following general form:

```
Premise  P1.
Premise  P2.
. . .
Premise  PN.
-----------------
Therefore, Conclusion.
```

An inference rule must have the property that the conclusion must be true whenever all the premises are true.

Equivalently, the following expression must be a tautology:

(P1 and P2 and … and PN) → Conclusion

Justifying an Inference Rule

We may form the truth table for the corresponding expression:

P or Q.
not P.
---------------------- *Modus tollendo ponens*
therefore, Q.

P	Q	P or Q	not P	(P or Q) and (not P)	(P or Q) and (not P) → Q
T	T	T	F	F	T
T	F	T	F	F	T
F	T	T	T	T	T
F	F	F	T	F	T

Note that we actually only needed to construct the truth table for the case in which the antecedent of the implication was true (third row).

Why?

Justifying an Inference Rule

The last observation leads to a somewhat simpler verbal analysis:

P or Q.
not P.
--------------------- *Modus tollendo ponens*
therefore, Q.

Assume the premises are both true. (This is the only case in which the argument form could possibly be false.)

Then from the definition of "not", since "not P" is true, P must be false.

But if "P or Q" is true, by the definition of "or", P and Q cannot both be false.

Hence, Q must be true.

And so, if both premises are true then the conclusion must also be true.

Modus ponens

Here's one of the most basic inference rules (we assume that P and Q represent specific propositions):

if P then Q.
P.
------------------ *Affirming the antecedent (Modus ponens)*
therefore, Q.

For example, if are given that:

If Cirrus is a cat, then Cirrus is carnivorous.
Cirrus is a cat.

Then we may infer:

Cirrus is carnivorous.

Validity and Soundness

An argument is *valid* if the conclusion is true whenever the premises are true.

An argument is *sound* if it is valid and the premises are, in fact, true.

The following argument is valid, but not sound:

Premises:

If January has 35 days then January contains 5 Mondays.

January has 35 days.

Conclusion:

January contains 5 Mondays.

Propositional logic is largely concerned with the issue of how to form valid arguments. The soundness of an argument generally depends upon external (domain-specific) knowledge.

Validity

Note that whether an argument is valid has nothing whatsoever to do with whether its premises are true or its conclusion is true:

Premises:

If Socrates was bipedal, then Socrates was a philosopher.
Socrates was bipedal.

Conclusion (via Modus ponens or Affirming the antecedent):

Socrates was a philosopher.

Premises:

If Grace Hopper was Australian, then she was a computer scientist.
Grace Hopper was Australian.

Conclusion:

Grace Hopper was a computer scientist.

Premises:

If Socrates was from Rome, then Socrates spoke Latin.
Socrates was from Rome.

Conclusion:

Socrates spoke Latin.

A Few More Inference Rules

Again, assume that P and Q represent specific propositions:

if P then Q.
not Q.
--------------------- *Denying the consequent (Modus tollens)*
therefore, not P.

P or Q.
not P.
--------------------- *Disjunctive syllogism (Modus tollendo ponens)*
therefore, Q.

not not P.
--------------------- *Double negation*
therefore, P.

P and Q.
--------------------- *Conjunction elimination*
therefore, P.

Example

if P then Q.
if not P then Q.
----
therefore, Q.

Assume the premises are both true. Also, note that P must be either true or false (although we have no idea which is the case).

Now, if P is true, then Q is true from the first premise via *modus ponens*.

And, if P is false, then not P is true, and therefore Q is true from the second premise via *modus ponens*.

Therefore, no matter whether P is true or P is false, Q must be true.

Why do we care?

If we want to derive results from a collection of alleged facts (premises) that we possess, we must understand what will guarantee that out results do, in fact, follow from the premises we are given.

Natural languages, especially English, promote using grammatical constructs that lend themselves to obscuring the underling logic, or lack thereof.

And, in any language, there may be implicit assumptions that are required to justify the stated conclusions. If those assumptions are false, then the argument is unsound, even if it is valid (and appears to be sensible when read without a full understanding of the true logic).

Consider the exchange quoted on the following slide…

Why do we care?

This was taken from an episode of a TV show:

Bob Murch, spirit board collector:
"There's been thousands of years of accounts of ghosts and hauntings, and if those are true, you know, surely a spirit board can work."

Penn Jillette:
"So, if those aren't true, a spirit board can't work? Cool!"

Source: Penn & Teller, "Ouija Boards/Near Death Experiences", *B.S.!*

Is there a flaw in Penn Jillette's logic?

How can you be sure?

How can you convince anyone of your conclusion?

Why do we care?

This was taken from the Roanoke Times for January 22, 2011:

Lincoln said the war was about the Union

Re: "Slavery was central to Civil War," Jan. 10 commentary:

When I read this essay, it became apparent the author is either a Yankee or someone boasting about his background in history. Maybe both, for him to go out of the way to put down the South as he did.

Just one question: If slavery was so central to causing the war to prevent Southern independence, why did President Lincoln fire Gen. John Fremont for declaring slaves would be set free after the Battle of Wilson's Creek in August 1861 and make the statement: "This war is being fought for a great national idea, the Union, and the general should not have dragged the Negro into it"?

Is the writer's logic valid? Is it sound?

Are there hidden assumptions (premises)?

Why do we care?

This was taken from Aristotle's *Politics*:

> Every state is a community of some kind, and every community is established with a view to some good; for mankind always act in order to obtain that which they think good.
>
> But, if all communities aim at some good, the state or political community, which is the highest of all, and which embraces all the rest, aims at good in a greater degree than any other, and at the highest good.

Is the writer's logic valid? Is it sound?

Are there hidden assumptions (premises)?

Example

This problem was taken from the *Puzzlers' Paradise* website:

Zookeeper George was in charge of feeding all of the animals in the morning. He had a regular schedule that he followed every day. Can you figure it out from the clues?

P1: Feedings begin at 6:30 am.
P2: A feeding takes 15 minutes.
P3: The last feeding begins no later than 7:30 am.
P4: The giraffes were fed before the zebras but after the monkeys.
P5: The bears were fed 15 minutes after the monkeys.
P6: The lions were fed after the zebras.

A carefully-reasoned analysis will provide a sequence of well-explained inferences, reaching a specific conclusion.

Example

P4: The giraffes were fed before the zebras but after the monkeys.

By Conjunction Elimination, P4 implies:

- **I1: Monkeys are fed before Giraffes.**
- **I2: Giraffes are fed before Zebras.**

P1: Feedings begin at 6:30 am.
P2: A feeding takes 15 minutes.

P2 and I1 imply

- **I3: Giraffes are fed at least 15 minutes after Monkeys.**

I3 and P1 imply

- **I4: Giraffes are fed no earlier than 6:45 am.**

Example

P4: The giraffes were fed before the zebras but after the monkeys.

By Conjunction Elimination, P4 implies:

I1: Monkeys are fed before Giraffes.

I2: Giraffes are fed before Zebras.

P1: Feedings begin at 6:30 am.

P2: A feeding takes 15 minutes.

P2 and I2 imply

I5: Zebras are fed at least 15 minutes after Giraffes.

I5 and I4 (Giraffes are fed no earlier than 6:45 am.) imply

I6: Zebras are fed no earlier than 7:00 am.

Example

P5: The bears were fed 15 minutes after the monkeys.

P1, P2, P5 and I3 imply
I7: Giraffes are fed at least 30 minutes after Monkeys.

I7 and P1 imply
I8: Giraffes are fed no earlier than 7:00 am.

I8 and I2 imply
I9: Zebras are fed no earlier than 7:15 am.

Example

P2: A feeding takes 15 minutes.
P6: The lions were fed after the zebras.
I9: Zebras are fed no earlier than 7:15 am.

P2, P6 and I6 imply
I10: Lions are fed no earlier than 7:30 am.
P3 and I10 imply
I11: Lions are fed at 7:30 am.

I3 – I7 imply a linear ordering:
I12: Monkeys precede Bears, which precede Giraffes, which precede Zebras, which precede Lions.

Example

P1 - P3 imply

I13: There are five possible feeding times:

6:30, 6:45, 7:00, 7:15 and 7:30.

So I12 and I13 imply that

George feeds Monkeys first at 6:30,

then Bears at 6:45,

then Giraffes at 7:00,

then Zebras at 7:15,

and finally Lions at 7:30.

Quantification

Quantifier --- a type of determiner, such as *all* or *some* or *many*, that indicates quantity.

All men are mortal.

Every prime integer larger than 2 is odd.

Some headlines exhibit ambiguity.

Most mammals are terrestrial rather than aquatic.

A few birds are naturally flightless.

Quantification: Domains

When a quantifier is used, there should usually be an associated *domain*; that is, we should know the collection of "things" that are the object of the quantifier.

Some student correctly answered every question ~~on the assessment test~~.

In the absence of a specific domain, the sentence may be ambiguous or meaningless.

Every x is prime.

Quantification: Order Matters

Every question on the assessment test was answered correctly by some student in the class.

Some student in the class correctly answered every question on the assessment test.

For every question on the assessment test, there exists a student in the class who answered that question correctly.

There exists a student in the class who answered every question on the assessment test correctly.

Quantification: Order Matters

For every integer x, there is some integer y such that x > y.

There is some integer y such that, for every integer x, x > y.

Verification: existential quantifiers

In order to show that an existentially-quantified statement is true, we merely must find some value in the domain for which the statement is true:

> There is an integer that equals twice the sum of its digits.

Consider 18.

Of course, finding a confirming example is not always trivial.

Verification: universal quantifiers

To show a universally-quantified statement is true, we must verify that the statement is true for every value in the domain:

> Every sentence on this slide contains an odd number of words.

Consider each sentence and count the words.

Of course, checking every value is not always possible.

Verification: universal quantifiers

In order to show that a universally-quantified statement is true, we must verify that the statement is true for every value in the domain:

For every even integer m, m*m + 1 is odd.

There are infinitely many even integers, so we cannot check them all.

Instead, we must make a formal argument (proof)… but that will be later.

Falsification: universal quantifiers

In order to show that a universally-quantified statement is false, we only need to find one instance of values from the domain for which the statement is false:

Every prime integer x is odd.

This is false because 2 is a prime integer and 2 is not odd.

Falsification: existential quantifiers

In order to show that an existentially-quantified statement is false, we must show that for every value from the domain, the statement is false:

> There is a volume in the 1980 edition of the Encyclopedia Britannica that has less than 100 pages.

I can show this is false by examining my copy of the 1980 edition of the Encyclopedia Britannica and verifying that every volume has (far) more than 100 pages… assuming that all copies of this edition are the same, which seems a fair assumption.

Falsification: existential quantifiers

But sometimes the domain is not finite:

> There is an integer larger than 10 that is equal to the product of its digits.

Now, this is harder.

We obviously cannot check every value in the domain.

This would call for a formal proof that the assumption such an integer exists implies some contradiction…

… or perhaps the statement is, in fact, true.

(A quick check with a short C program verifies no integer less than one billion has the property… but that proves nothing.)

Quantification: Negation

What is the negation of:

Every prime integer is odd.

Clearly, the negation must deny the assertion that every prime integer has the stated property.

That means the negation must merely claim that there is some prime integer that does not have the stated property:

There is a prime integer that is not odd.

Quantification: Negation

What is the negation of:

Some three-digit integer equals the product of its digits.

Clearly, the negation must deny the assertion that there is an integer with three digits has the stated property.

That means the negation must merely claim that there is no integer with three digits that does have the stated property:

No three-digit integer equals the product of its digits.

Alternatively:

Every three-digit integer is unequal to the product of its digits.

Quantification: Negation and the Domain

Note that negating does not alter the domain:

All dogs are endotherms.

Some dogs are not endotherms.

You cannot deny the truth of the first statement by talking about the wrong set of things:

Some iguanas are not endotherms.

Quantification: Formal Negation

So:

"not (every x in D has property P)" is
"some x in D does not have property P"

and

"not (some x in D has property P)" is
"no x in D has property P"

Examples

Of course, it gets more interesting; negate these:

For every natural number N, there is a natural number M such that M < N.

There is a natural number M such that, for every natural number N, M ≤ N.

For every integer N, if N is a multiple of 2 then $\log_2(N)$ is an integer.

For every real number ε > 0, there is a real number N > 0 such that whenever x is a real number such that x > N, 1/x < ε.

Answers

The negations:

There is a natural number N such that for every natural number M, M ≥ N.

For every natural number M, there is a natural number N such that M > N.

There is an integer N such that N is a multiple of 2 and $\log_2(N)$ is not an integer.

There is a real number ε > 0 such that, for every real number N > 0, there is a real number x such that x > N, but 1/x ≥ ε.