Network Flow

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Maximum Flow and Minimum Cut

▶ Two rich algorithmic problems.

▶ Fundamental problems in combinatorial optimization.

▶ Beautiful mathematical duality between flows and cuts.

▶ Numerous non-trivial applications:

▶ Bipartite matching.

▶ Data mining.

▶ Project selection.

▶ Airline scheduling.

▶ Baseball elimination.

▶ Image segmentation.

▶ Network connectivity.

▶ Open-pit mining.

▶ Network reliability.

▶ Distributed computing.

▶ Egalitarian stable matching.

▶ Security of statistical data.

▶ Network intrusion detection.

▶ Multi-camera scene reconstruction.

▶ Gene function prediction.

History
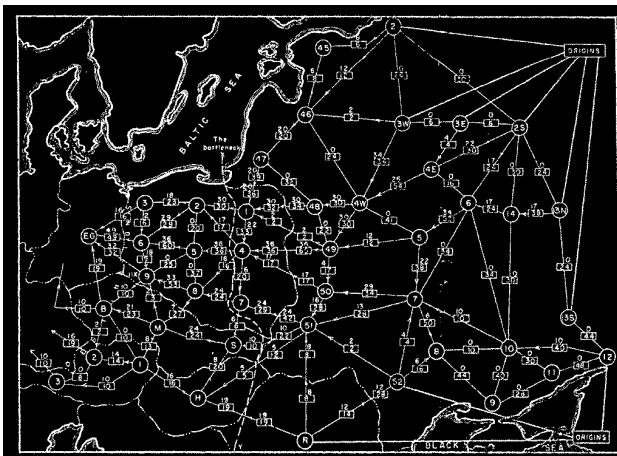
(Soviet Rail Network, Tolstoi, 1930; Harris and Ross, 1955; Alexander Schrijver, *Math Programming*, 91: 3, 2002.)

Flow Networks

▶ Use directed graphs to model *transporation networks*:
  ▶ edges carry traffic and have capacities.
  ▶ nodes act as switches.
  ▶ *source* nodes generate traffic, *sink* nodes absorb traffic.

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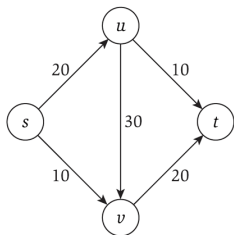 ▶ *source* nodes generate traffic, *sink* nodes absorb traffic.

Figure 7.2 A flow network, with source s and sink t. The numbers next to the edges are the capacities.

▶ A *flow network* is a directed graph $G(V, E)$

  ▶ Each edge $e \in E$ has a capacity $c(e) > 0$.
  ▶ There is a single *source* node $s \in V$.
  ▶ There is a single *sink* node $t \in V$.
  ▶ Nodes other than s and t are *internal*.

Defining Flow

▶ In a flow network $G(V, E)$, an *s-t flow* is a function $f : E \rightarrow \mathbb{R}^+$ such that

(i) (*Capacity conditions*) For each $e \in E$, $0 \leq f(e) \leq c(e)$.

(ii) (*Conservation conditions*) For each internal node v,

$$\sum_{e \text{ into } v} f(e) = \sum_{e \text{ out of } v} f(e)$$

▶ The *value* of a flow is $\nu(f) = \sum_{e \text{ out of } s} f(e)$.

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- The *value* of a flow is $\nu(f) = \sum_{e \text{ out of } s} f(e)$.
- Useful notation:
  - $f^{\text{out}}(v) = \sum_{e \text{ out of } v} f(e)$, $f^{\text{in}}(v) = \sum_{e \text{ into } v} f(e)$.
  - For $S \subseteq V$, $f^{\text{out}}(S) = \sum_{e \text{ out of } S} f(e)$, $f^{\text{in}}(S) = \sum_{e \text{ into } S} f(e)$.

Maximum-Flow Problem

MAXIMUM FLOW
INSTANCE: A flow network G
SOLUTION: The flow with largest value in G

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► Assumptions:
  1. No edges enter s, no edges leave t.
  2. There is at least one edge incident on each node.
  3. All edge capacities are integers.

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Figure 7.3 (a) The network of Figure 7.2. (b) Pushing 20 units of flow along the path s, u, v, t. (c) The new kind of augmenting path using the edge (u, v) backward.

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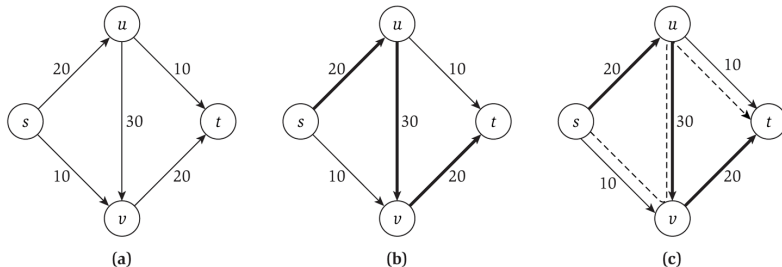   1. Start with zero flow along all edges (Figure 7.3(a)).
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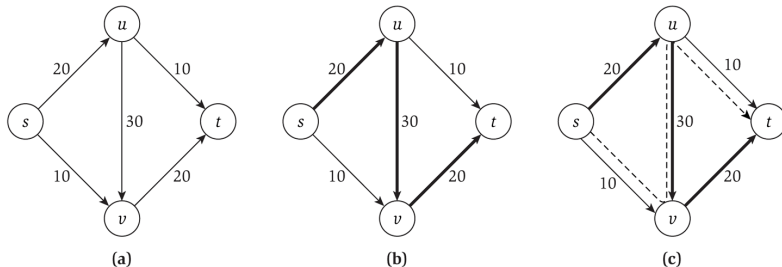   (Figure 7.3(b)).
  3. Idea: Push flow along edges with leftover capacity and undo flow on
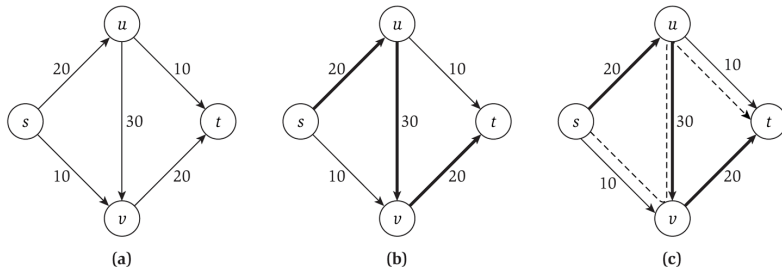   edges already carrying flow.

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Residual Graph

▶ Given a flow network $G(V, E)$ and a flow f on G, the *residual graph* G_f of G with respect to f is a directed graph such that

(i) (*Nodes*) G_f has the same nodes as G.

(ii) (*Forward edges*) For each edge $e = (u, v) \in E$ such that $f(e) < c(e)$, G_f contains the edge (u, v) with a *residual capacity* $c(e) - f(e)$.

(iii) (*Backward edges*) For each edge $e \in E$ such that $f(e) > 0$, G_f contains the edge $e' = (v, u)$ with a capacity $f(e)$.

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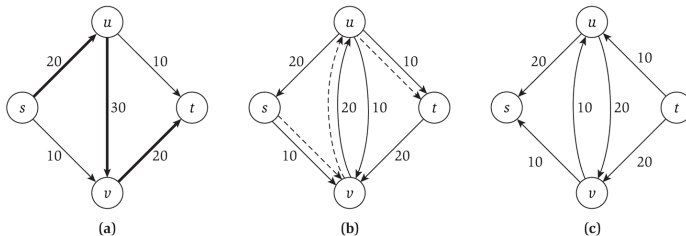
Augmenting Paths in a Residual Graph

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- ▶ *bottleneck*(P, f) is the minimum residual capacity of any edge in P.
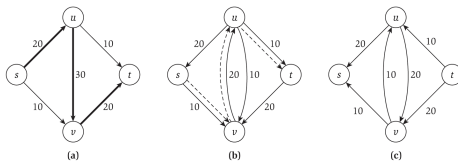
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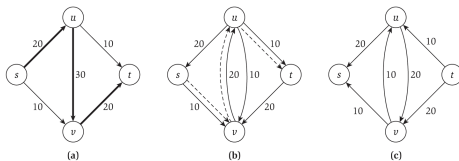
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▶ Let P be a simple s-t path in G_f.
▶ *bottleneck*(P, f) is the minimum residual capacity of any edge in P.
▶ The following operation `augment(f, P)` yields a new flow f' in G:

```
augment(f, P)
  Let b = bottleneck(P, f)
  For each edge (u, v) ∈ P
    If e = (u, v) is a forward edge then
      increase f(e) in G by b
    Else ((u, v) is a backward edge, and let e = (v, u))
      decrease f(e) in G by b
    Endif
  Endfor
  Return(f)
```

Correctness of augment(f, P)

- A simple s-t path in the residual graph is an *augmenting path*.
- Let f' be the flow returned by augment(f, P).
- Claim: f' is a flow. Verify capacity and conservation conditions.

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      $c(e) \geq f(e) \geq f'(e) = f(e) - \text{bottleneck}(P, f) \geq f(e) - f(e) = 0$.
  - Conservation condition on internal node $v \in P$. Four cases to work out.

Ford-Fulkerson Algorithm

Max-Flow

```
Initially f(e) = 0 for all e in G
While there is an s-t path in the residual graph G_f
    Let P be a simple s-t path in G_f
    f' = augment(f, P)
    Update f to be f'
    Update the residual graph G_f to be G_f'
Endwhile
Return f
```
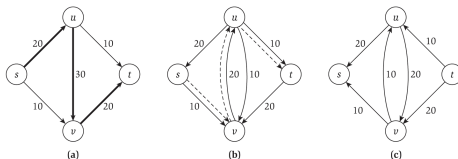

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Analysis of the Ford-Fulkerson Algorithm

► Running time
  ► Does the algorithm terminate?
  ► If so, how many loops does the algorithm take?

► Correctness: if the algorithm terminates, why does it output a maximum flow?

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- ▶ Claim: Algorithm runs in $O(mC)$ time.

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- Can we characterise the magnitude of the flow in terms of the structure of the graph? For example, for every flow f, $\nu(f) \leq C = \sum_{e\text{out of } s} c(e)$.
- Is there a better bound?

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▶ Is there a better bound?

▶ Idea: An *s-t cut* is a partition of V into sets A and B such that $s \in A$ and $t \in B$.

   ▶ *Capacity* of the cut (A, B) is $c(A, B) = \sum_{e \text{ out of } A} c(e)$.
   ▶ Intuition: For every flow f, $\nu(f) \leq c(A, B)$.

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    - $\sum_{v \in A} \left(f^{\text{out}}(v) - f^{\text{in}}(v) \right) = \sum_{e \text{ out of } A} f(e) - \sum_{e \text{ into } A} f(e) = f^{\text{out}}(A) - f^{\text{in}}(A)$.
- Corollary: $\nu(f) = f^{\text{in}}(B) - f^{\text{out}}(B)$.

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- Corollary: $\nu(f) = f^{\text{in}}(B) - f^{\text{out}}(B)$.
- $\nu(f) \leq c(A, B)$.

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▶ Corollary: $\nu(f) = f^{\text{in}}(B) - f^{\text{out}}(B)$.
▶ $\nu(f) \leq c(A, B)$. $\nu(f) = f^{\text{out}}(A) - f^{\text{in}}(A) \leq f^{\text{out}}(A) = \sum_{e \text{ out of } A} f(e) \leq \sum_{e \text{ out of } A} c(e) = c(A, B)$.

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- Let f be any s-t flow and (A, B) any s-t cut. We proved $\nu(f) \leq c(A, B)$.
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- Corollary: The maximum flow is at most the smallest capacity of a cut.

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- Very strong statement: The value of every flow is at most the capacity of any cut.
- Corollary: The maximum flow is at most the smallest capacity of a cut.
- Question: Is the reverse true? Is the smallest capacity of a cut at most the maximum flow?
- Answer: Yes, and the Ford-Fulkerson algorithm computes this cut!

Flows and Cuts

- Let $\bar{f}$ denote the flow computed by the Ford-Fulkerson algorithm.
- Enough to show $\exists$ s-t cut (A^*, B^*) such that $\nu(\bar{f}) = c(A^*, B^*)$.
- When the algorithm terminates, the residual graph has no s-t path.

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- When the algorithm terminates, the residual graph has no s-t path.
- Claim: If f is an s-t flow that G_f has no s-t path, then there is an s-t cut (A^*, B^*) such that $\nu(f) = c(A^*, B^*)$.

Proof of Claim Relating Flows to Cuts

▶ Claim: f is an s-t flow, G_f has no s-t path $\Rightarrow \exists$ s-t cut (A^*, B^*), $\nu(f) = c(A^*, B^*)$.

  ▶ $A^* =$ set of nodes reachable from s in G_f, $B = V - A^*$.
  ▶ Claim: (A, B) is an s-t cut.
  ▶ Claim: If $e = (u, v)$ such that $u \in A^*$, $v \in B^*$, then

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   ► Claim: (A, B) is an s-t cut.

   ► Claim: If $e = (u, v)$ such that $u \in A^*$, $v \in B^*$, then $f(e) = c(e)$.

   ► Claim: If $e = (u, v)$ such that $u \in B^*$, $v \in A^*$, then

Proof of Claim Relating Flows to Cuts

- ▶ Claim: f is an s-t flow, G_f has no s-t path $\Rightarrow \exists$ s-t cut (A^*, B^*), $\nu(f) = c(A^*, B^*)$.
  - ▶ $A^* =$ set of nodes reachable from s in G_f, $B = V - A^*$.
  - ▶ Claim: (A, B) is an s-t cut.
  - ▶ Claim: If $e = (u, v)$ such that $u \in A^*$, $v \in B^*$, then $f(e) = c(e)$.
  - ▶ Claim: If $e = (u, v)$ such that $u \in B^*$, $v \in A^*$, then $f(e) = 0$.
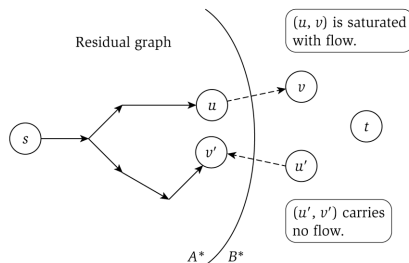 - ▶ Claim: $\nu(f) = c(A^*, B^*)$.

Figure 7.5 The (A^*, B^*) cut in the proof of (7.9).

Max-Flow Min-Cut Theorem

- The flow $\bar{f}$ computed by the Ford-Fulkerson algorithm is a maximum flow.
- Given a flow of maximum value, we can compute a minimum s-t cut in $O(m)$ time.
- In every flow network, there is a flow f and a cut (A, B) such that $\nu(f) = c(A, B)$.

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- In every flow network, there is a flow f and a cut (A, B) such that $\nu(f) = c(A, B)$.
- Max-Flow Min-Cut Theorem: in every flow network, the maximum value of an s-t flow is equal to the minimum capacity of an s-t cut.
- Corollary: If all capacities in a flow network are integers, then there is a maximum flow f where every flow value $f(e)$ is an integer.

Real-Valued Capacities

- If capacities are real-valued, Ford-Fulkerson algorithm may not terminate!
- But Max-Flow Min-Cut theorem is still true. Why?

Bad Augmenting Paths

Figure 7.6 Parts (a) through (d) depict four iterations of the Ford-Fulkerson Algorithm using a bad choice of augmenting paths: The augmentations alternate between the path P_1 through the nodes s, u, v, t in order and the path P_2 through the nodes s, v, u, t in order.
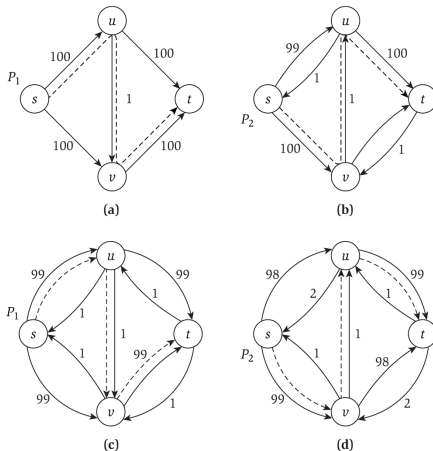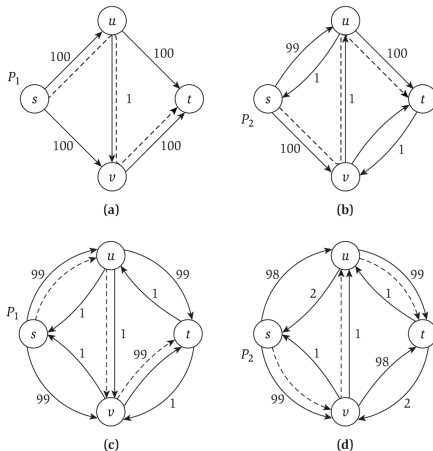
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Improving Ford-Fulkerson Algorithm

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▶ Idea: decrease number of iterations by picking s-t path with bottleneck edge of largest capacity.

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▶ Modified idea: Maintain a *scaling parameter* Δ and choose only augmenting paths with bottleneck capacity at least Δ.

Improving Ford-Fulkerson Algorithm

- Bad case for Ford-Fulkerson algorithm is when the bottleneck edge is the augmenting path has a low capacity.
- Idea: decrease number of iterations by picking s-t path with bottleneck edge of largest capacity. Computing this path can slow down each iteration considerably.
- Modified idea: Maintain a *scaling parameter* Δ and choose only augmenting paths with bottleneck capacity at least Δ.
- $G_f(\Delta)$: residual network restricted to edges with residual capacities at least Δ.

Scaling Max-Flow Algorithm

```
Scaling Max-Flow
   Initially f(e) = 0 for all e in G
   Initially set Δ to be the largest power of 2 that is no larger
         than the maximum capacity out of s: Δ ≤ max_{e out of s} c_e
      While Δ ≥ 1
         While there is an s-t path in the graph G_f(Δ)
            Let P be a simple s-t path in G_f(Δ)
            f' = augment(f, P)
            Update f to be f' and update G_f(Δ)
         Endwhile
         Δ = Δ/2
      Endwhile
Return f
```

Correctness of the Scaling Max-Flow Algorithm

▶ Flow and residual capacities are integer valued throughout.

▶ When $\Delta = 1$, $G_f(\Delta)$ and G_f are identical.

▶ Therefore, the scaling algorithm terminates, the flow is a maximum flow.

Running time of the Scaling Max-Flow Algorithm

- ▶ *Δ-scaling phase*: one iteration of the outer loop of the algorithm, with Δ fixed.
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▶ Claim: the value of the maximum flow is at most $\nu(f) + m\Delta$.

▶ Claim: the number of augmentations in a Δ-scaling phase is $\leq 2m$.

▶ Claim: the running time of the scaling max-flow algorithm is $O(m^2 \log C)$.

Other Maximum Flow Algorithms

- Running time of the Ford-Fulkerson algorithm is $O(mC)$, which is *pseudo-polynomial*: polynomial in the magnitudes of the numbers in the input.

- Scaling algorithm runs in time polynomial in the size of the input (the graph and the number of bits needed to represent the capacities).

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- Edmonds-Karp, Dinitz: choose augmenting path to be the shortest path in G_f (use breadth-first search). Algorithm runs in $O(mn)$ iterations.

- Improved algorithms take time $O(mn \log n)$, $O(n^3)$, etc.

- Chapter 7.4: Preflow-push max-flow algorithm that is not based on augmenting paths. Runs in $O(n^2 m)$ or $O(n^3)$ time.