Divide and Conquer Algorithms

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Divide and Conquer Algorithms

- Study three divide and conquer algorithms:
  - Counting inversions.
  - Finding the closest pair of points.
  - Integer multiplication.
- First two problems use clever conquer strategies.
- Third problem uses a clever divide strategy.

Motivation

- Collaborative filtering: match one user's preferences to those of other users.
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- Fundamental question: how do we compare a pair of rankings?
- Suggestion: two rankings are very similar if they have few inversions.
  - Assume one ranking is the ordered list of integers from 1 to n.
  - The other ranking is a permutation $a_1, a_2, \ldots, a_n$ of the integers from 1 to n.
  - The second ranking has an *inversion* if there exist i, j such that $i < j$ but $a_i > a_j$.
  - The number of inversions s is a measure of the difference between the rankings.
- Question also arises in statistics: *Kendall's rank correlation* of two lists of numbers is $1 - 2s/\left(n(n-1)\right)$.

Counting Inversions

COUNT INVERSIONS
INSTANCE: A list $L = x_1, x_2, \ldots, x_n$ of distinct integers between 1 and n.
SOLUTION: The number of pairs $(i, j), 1 \leq i < j \leq n$ such $a_i > a_j$.

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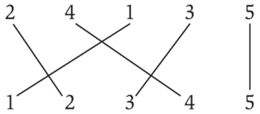

Figure 5.4 Counting the number of inversions in the sequence 2, 4, 1, 3, 5. Each crossing pair of line segments corresponds to one pair that is in the opposite order in the input list and the ascending list—in other words, an inversion.

Counting Inversions: Algorithm

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- Sorting removes all inversions in $O(n \log n)$ time. Can we modify the Mergesort algorithm to count inversions?
- Candidate algorithm:
  1. Partition L into two lists A and B of size $n/2$ each.
  2. Recursively count the number of inversions in A.
  3. Recursively count the number of inversions in B.
  4. Count the number of inversions involving one element in A and one element in B.

Counting Inversions: Conquer Step

- Given lists $A = a_1, a_2, \ldots, a_m$ and $B = b_1, b_2, \ldots b_m$, compute the number of pairs a_i and b_j such $a_i > b_j$.

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- Key idea: problem is much easier if A and B are sorted!

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- Key idea: problem is much easier if A and B are sorted!
- MERGE-AND-COUNT procedure:

  Maintain a *current* pointer for each list.
  Maintain a variable *count* initialised to 0.
  Initialise each pointer to the front of the list.
  While both lists are nonempty:
    Let a_i and b_j be the elements pointed to by the *current* pointers.
    Append the smaller of the two to the output list.
    If b_j is the smaller, increment *count* by the number of elements remaining in A.
    Advance the current pointer in the list that the smaller element belonged to.
  EndWhile
  Append the rest of the non-empty list to the output.
  Return *count* and the merged list.

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  EndWhile
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  Return *count* and the merged list.
- Running time of this algorithm is $O(m)$.

Counting Inversions: Final Algorithm

```
Sort-and-Count(L)
  If the list has one element then
      there are no inversions
  Else
      Divide the list into two halves:
         A contains the first ⌈n/2⌉  elements
         B contains the remaining ⌊n/2⌋ elements
      (r_A, A) = Sort-and-Count(A)
      (r_B, B) = Sort-and-Count(B)
      (r, L) = Merge-and-Count(A, B)
   Endif
   Return r = r_A + r_B + r, and the sorted list L
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- Running time $T(n)$ of the algorithm is $O(n \log n)$ because $T(n) \leq 2T(n/2) + O(n)$.

Computational Geometry

- Algorithms for geometric objects: points, lines, segments, triangles, spheres, polyhedra, ldots.
- Started in 1975 by Shamos and Hoey.
- Problems studied have applications in a vast number of fields: ecology, molecular biology, statistics, computational finance, computer graphics, computer vision, ...

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INSTANCE: A set P of n points in the plane

SOLUTION: The pair of points in P that are the closest to each other.

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- At first glance, it seems any algorithm must take $\Omega(n^2)$ time.
- Shamos and Hoey figured out an ingenious $O(n \log n)$ divide and conquer algorithm.

Closest Pair: Set-up

- Let $P = \{p_1, p_2, \ldots, p_n\}$ with $p_i = (x_i, y_i)$.
- Use $d(p_i, p_j)$ to denote the Euclidean distance between p_i and p_j.
- Goal: find the pair of points p_i and p_j that minimise $d(p_i, p_j)$.

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- How do we solve the problem in 1D? Sort: closest pair must be adjacent in the sorted order.
- The idea does not work in 2D.

Closest Pair: Algorithm Skeleton

1. Divide P into two sets Q and R of $n/2$ points such that each point in Q has x-coordinate less than any point in R.
2. Recursively compute closest pair in Q and in R, respectively.

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3. Let δ_1 be the distance computed for Q, δ_2 be the distance computed for R, and $\delta = \min(\delta_1, \delta 2)$.
4. Compute pair (q, r) of points such that $q \in Q$, $r \in R$, $d(q, r) < \delta$ and $d(q, r)$ is the smallest possible.

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   - How do we implement this step in $O(n)$ time?

Closest Pair: Conquer Step

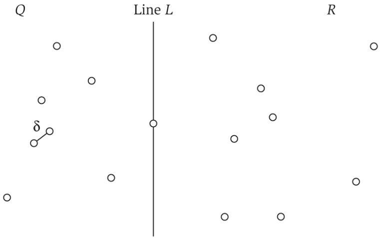

Figure 5.6 The first level of recursion: The point set P is divided evenly into Q and R by the line L, and the closest pair is found on each side recursively.

- Line L passes through right-most point in Q.

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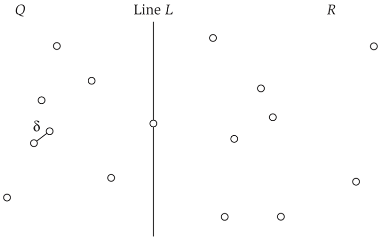

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- Claim: If there exist $q \in Q$, $r \in R$ such that $d(q, r) < \delta$, then q and r are both within distance δ of L.

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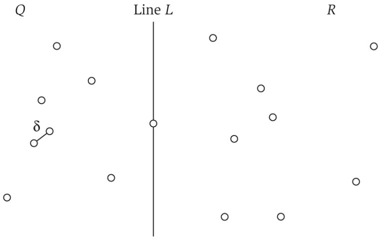

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- Claim: If there exist $q \in Q$, $r \in R$ such that $d(q, r) < \delta$, then q and r are both within distance δ of L.
- Let S be the set of points within distance δ of L and let S_y denote these points sorted by increasing y-coordinate.
- Claim: There exist $q \in Q$, $r \in R$ such that $d(q, r) < \delta$ if and only if there exist $s, s' \in S$ such that $d(s, s') < \delta$.

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- Intuition: if there are "too many" points in S that are closer than δ to each other, then there must be a pair in Q or in R that are less than δ apart.

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- For a point $s \in S$, let s_y denote its y-coordinate.
- Converse of the claim: If there exist $s, s' \in S$ such that s' appears 16 or more indices after s in S_y, then $s'_y - s_y \geq \delta$.

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- Idea behind the proof: pack the plane with squares, argue that each square contains at most one point.

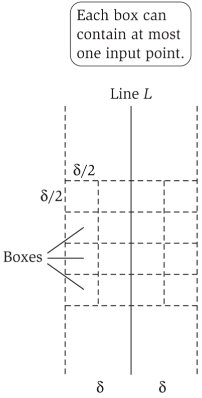

Figure 5.7 The portion of the plane close to the dividing line L, as analyzed in the proof of (5.10).

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  Construct P_x and P_y  (O(n log n) time)
  (p*_0, p*_1) = Closest-Pair-Rec(P_x,P_y)

Closest-Pair-Rec(P_x, P_y)
  If |P| ≤ 3 then
    find closest pair by measuring all pairwise distances
  Endif

  Construct Q_x, Q_y, R_x, R_y (O(n) time)
  (q*_0,q*_1) = Closest-Pair-Rec(Q_x, Q_y)
  (r*_0,r*_1) = Closest-Pair-Rec(R_x, R_y)

  δ = min(d(q*_0,q*_1), d(r*_0,r*_1))
  x* = maximum x-coordinate of a point in set Q
  L = {(x,y) : x = x*}
  S = points in P within distance δ of L.

  Construct S_y (O(n) time)
  For each point s ∈ S_y, compute distance from s
     to each of next 15 points in S_y
     Let s, s' be pair achieving minimum of these distances
     (O(n) time)

  If d(s,s') < δ then
     Return (s,s')
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```
                1100
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              ------
                1100
   12          0000
 × 13         1100
 ----        1100
   36        --------
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 ----
  156

  (a)          (b)
```

Figure 5.8 The elementary-school algorithm for multiplying two integers, in (a) decimal and (b) binary representation.

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- Multiply two n-digit integers.
- Result has at most $2n$ digits.
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$$
\begin{aligned}
xy &= (x_1 2^{n/2} + x_0)(y_1 2^{n/2} + y_0) \\
&= x_1 y_1 2^n + (x_1 y_0 + x_0 y_1) 2^{n/2} + x_0 y_0.
\end{aligned}
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- Each of x_1, x_0, y_1, y_0 has $n/2$ bits, so we can compute $x_1 y_1$, $x_1 y_0$, $x_0 y_1$, and $x_0 y_0$ recursively, and merge the answers in $O(n)$ time.

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$$\begin{aligned} T(n) &\leq 4T(n/2) + cn \\ &\leq O(n^2) \end{aligned}$$

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  - $x_1 y_1 + (x_1 y_0 + x_0 y_1) + x_0 y_0 = (x_0 + x_1)(y_0 + y_1)$
  - Compute $x_1 y_1$, $x_0 y_0$ and $(x_0 + x_1)(y_0 + y_1)$ recursively and then compute $(x_1 y_0 + x_0 y_1)$ by subtraction.
  - We have three sub-problems of size $n/2$.
- What is the running time $T(n)$?

Improving the Algorithm

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$$\begin{aligned} T(n) &\leq 3T(n/2) + cn \\ &\leq O(n^{\log_2 3}) = O(n^{1.59}) \end{aligned}$$

Final Algorithm

```
Recursive-Multiply(x,y):
  Write x = x_1 · 2^{n/2} + x_0
        y = y_1 · 2^{n/2} + y_0
  Compute x_1 + x_0 and y_1 + y_0
  p = Recursive-Multiply(x_1 + x_0, y_1 + y_0)
  x_1y_1 = Recursive-Multiply(x_1, y_1)
  x_0y_0 = Recursive-Multiply(x_0, y_0)
  Return x_1y_1 · 2^n + (p − x_1y_1 − x_0y_0) · 2^{n/2} + x_0y_0
```