Greedy Graph Algorithms

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Graphs

- Model pairwise relationships (edges) between objects (nodes).
- *Undirected graph* $G = (V, E)$: set V of nodes and set E of edges, where $E \subseteq V \times V$. Elements of E are unordered pairs.
- *Directed graph* $G = (V, E)$: set V of nodes and set E of edges, where $E \subseteq V \times V$. Elements of E are ordered pairs.

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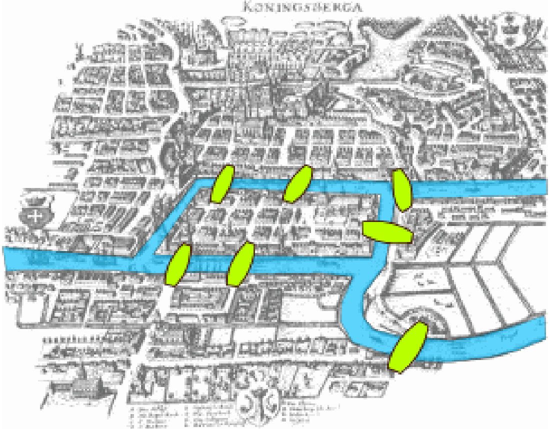

Shortest Path Problem

- $G(V, E)$ is a connected directed graph. Each edge e has a length $l_e \geq 0$.
- V has n nodes and E has m edges.
- *Length of a path* P is the sum of lengths of the edges in P.
- Goal is to determine the shortest path from some start node s to each nodes in V.
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Shortest Paths

INSTANCE: A directed graph $G(V, E)$, a function $l : E \rightarrow \mathbb{R}^+$, and a node $s \in V$

SOLUTION: A set $\{P_u, u \in V\}$, where P_u is the shortest path in G from s to u.

Dijkstra's Algorithm

- Maintain a set S of explored nodes: for each node $u \in S$, we have determined the length $d(u)$ of the shortest path from s to u.
- "Greedily" add a node v to S that is closest to s.

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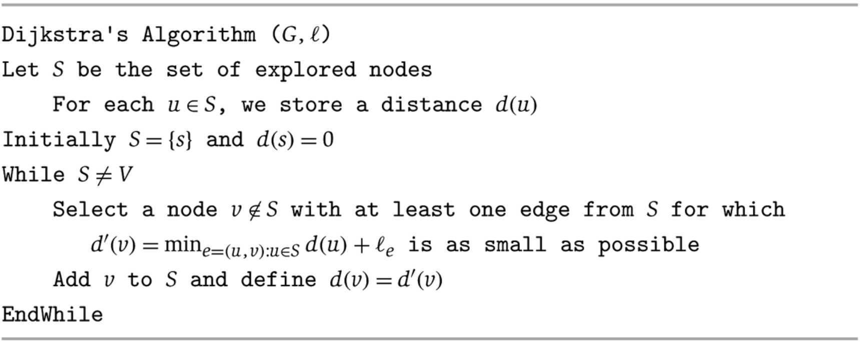

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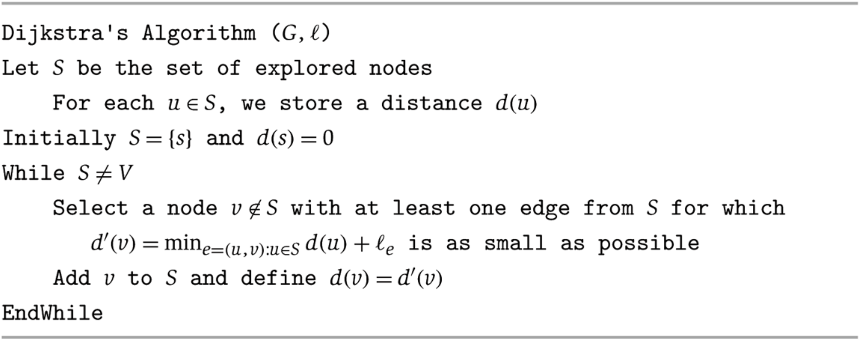

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- Can modify algorithm to compute the shortest paths themselves: record the predecessor u that minimises $d'(v)$.

Example of Dijkstra's Algorithm

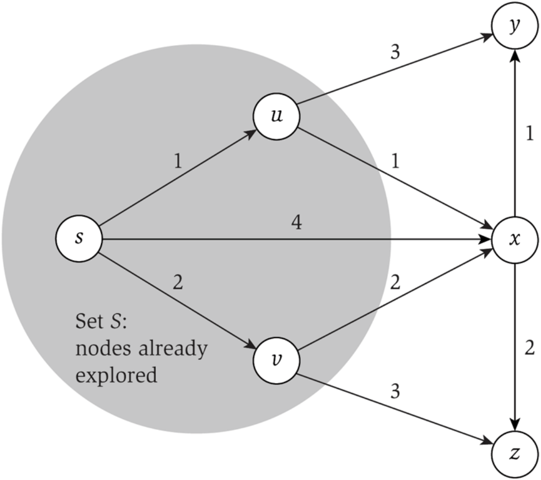

Figure 4.7 A snapshot of the execution of Dijkstra's Algorithm. The next node that will be added to the set S is x, due to the path through u.

Proof of Correctness

- Let P_u be the shortest path computed for a node u.
- Claim: P_u is the shortest path from s to u.
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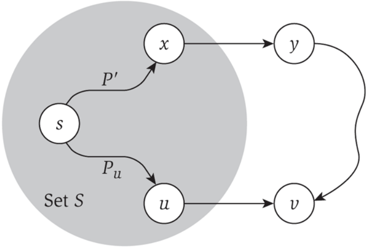


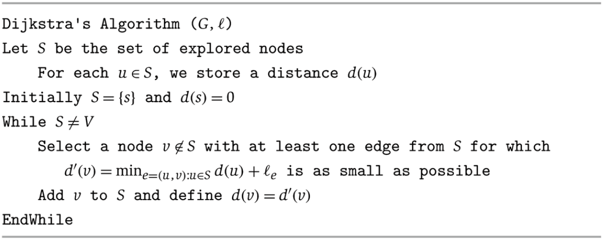
The alternate s–v path P through x and y is already too long by the time it has left the set S.

Figure 4.8 The shortest path P_v and an alternate s-v path P through the node y.

Comments about Dijkstra's Algorithm

- Algorithm cannot handle negative edge lengths. We will discuss the Bellman-Ford algorithm in a few weeks.
- Union of shortest paths output form a tree. Why?

Implementing Dijkstra's Algorithm



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- How many iterations are there of the while loop?

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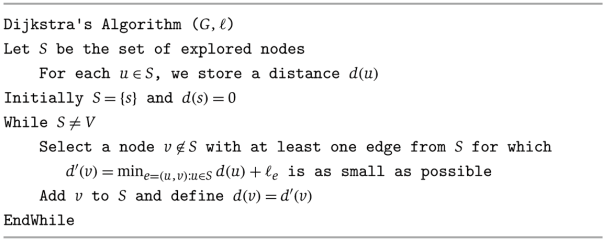

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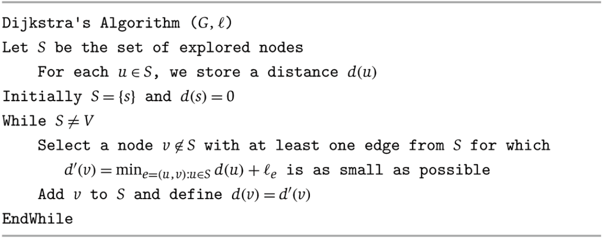

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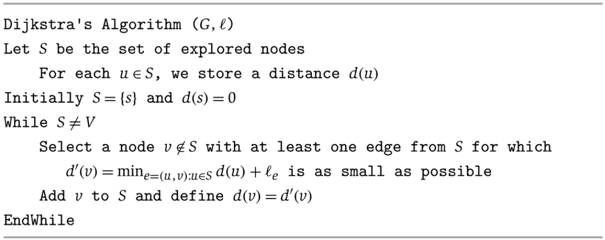

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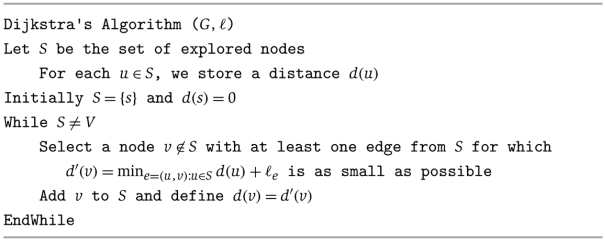

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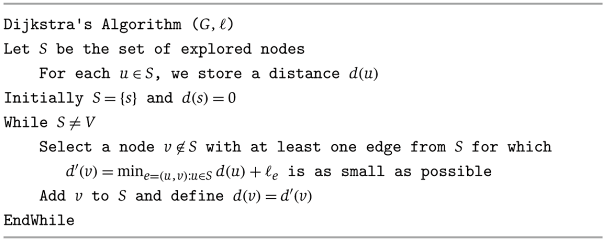
- Running time per iteration is $O(m)$, yielding an overall running time of $O(nm)$.

A Faster implementation of Dijkstra's Algorithm



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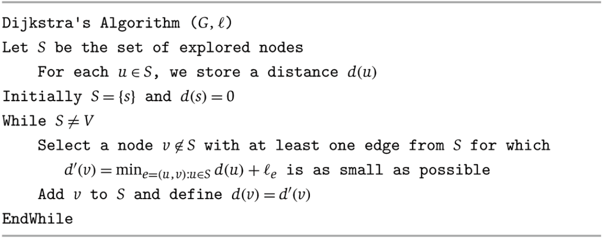

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- Store the minima $d'(v)$ for each node $v \in V - S$ in a priority queue.
- Determine the next node v to add to S using EXTRACTMIN.
- After adding v, for each neighbour w of v, compute $d(v) + l_{(v,w)}$.
- If $d(v) + l_{(v,w)} < d'(w)$, update w's key using CHANGEKEY.

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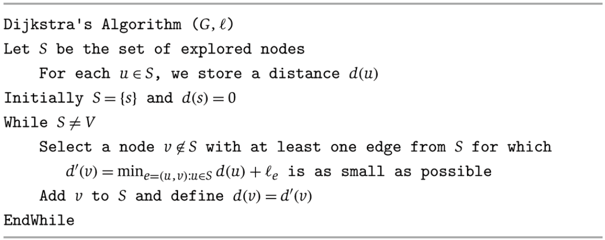

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- How many times are EXTRACTMIN and CHANGEKEY invoked?

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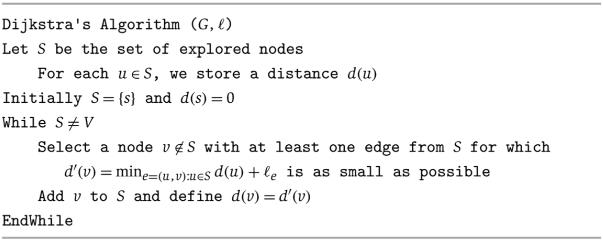

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- How many times are ExtractMin and ChangeKey invoked? $n - 1$ and m times, respectively.

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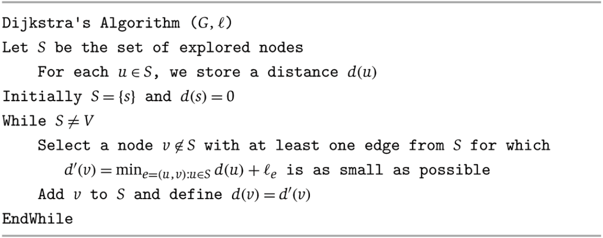

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- How many times are ExtractMin and ChangeKey invoked? $n - 1$ and m times, respectively. Total running time is $O(m \log n)$.

Network Design

- Connect a set of nodes using a set of edges with certain properties.
- Input is usually a graph and the desired network (the output) should use subset of edges in the graph.
- Example: connect all nodes using a cycle of shortest total length.

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Minimum Spanning Tree (MST)

- Given an undirected graph $G(V, E)$ with a cost $c_e > 0$ associated with each edge $e \in E$.
- Find a subset T of edges such that the graph (V, T) is connected and the cost $\sum_{e \in T} c_e$ is as small as possible.

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- Claim: If T is a minimum-cost solution to this network design problem then (V, T) is a tree.
- A subset T of E is a *spanning tree* of G if (V, T) is a tree.

Greedy Algorithm for the MST Problem

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  Increasing cost order Process edges in increasing order of cost. Discard an edge if it creates a cycle.

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Greedy Algorithm for the MST Problem

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  Increasing cost order Process edges in increasing order of cost. Discard an edge if it creates a cycle. Kruskal's algorithm

  Dijkstra-like Start from a node s and grow T outward from s: add the node that can be attached most cheaply to current tree. Prim's algorithm

  Decreasing cost order Delete edges in order of decreasing cost as long as graph remains connected. Reverse-Delete algorithm

- Which of these algorithms works? All of them!

Example of Prim's and Kruskal's Algorithms

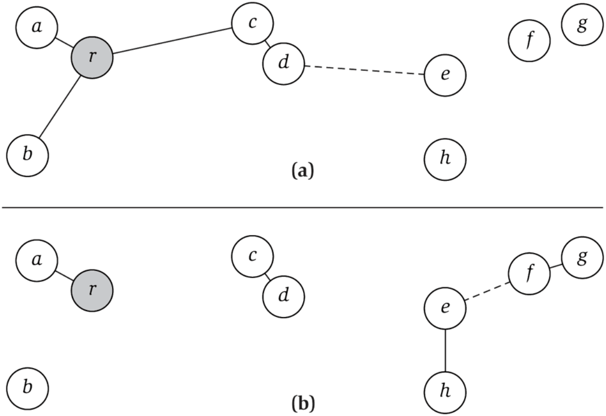

Figure 4.9 Sample run of the Minimum Spanning Tree Algorithms of (a) Prim and (b) Kruskal, on the same input. The first 4 edges added to the spanning tree are indicated by solid lines; the next edge to be added is a dashed line.

Graph Cuts

- A *cut* in a graph $G(V, E)$ is a set of edges whose removal disconnects the graph (into two or more connected components).
- Every set $S \subset V$ (S cannot be empty or the entire set V) has a corresponding cut: $\text{cut}(S)$ is the set of edges (v, w) such that $v \in S$ and $w \in V - S$.

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- $\text{cut}(S)$ is a cut because deleting the edges in $\text{cut}(S)$ disconnects S from $V - S$.

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- Assume all edge costs are distinct.
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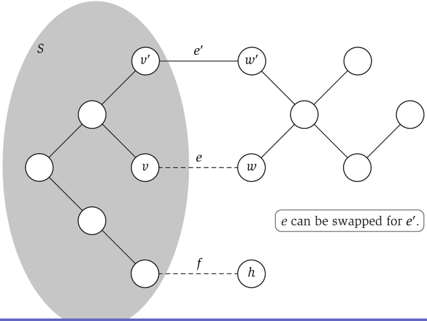
- Claim: every MST contains e.
- Proof: exchange argument. If a supposed MST T does not contain e, show that there is a tree with smaller cost than T that contains e.



e can be swapped for e'.

Using the Cut Property

- Let F be the set of all edges that satisfy the cut property.
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Using the Cut Property

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- Is the graph induced by F connected? Yes.
- Can the graph induced by F contain a cycle? No.
- How many edges can F contain? $n - 1$
- F is the unique MST (problem 8, Chapter 4 of textbook).
- Kruskal's and Prim's algorithms compute F efficiently.

Optimality of Kruskal's Algorithm

- Kruskal's algorithm:
  - Start with an empty set T of edges.
  - Process edges in E in increasing order of cost.
  - Add the next edge e to T only if adding e does not create a cycle. Discard e if it creates a cycle.
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- Claim: Kruskal's algorithm outputs an MST.
  1. For every edge e added, demonstrate the existence of S and $V - S$ such that e and S satisfy the cut property.
  2. Prove that the algorithm computes a spanning tree.

Optimality of Prim's Algorithm

- Prim's algorithm: Maintain a tree (S, U)
  - Start with an arbitrary node $s \in S$ and $U = \emptyset$.
  - Add the node v to S and the edge e to U that minimise

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- Let C be any cycle in G and let $e = (v, w)$ be the most expensive edge in C.
- Claim: e does not belong to any MST of G.
- Proof: exchange argument. If a supposed MST T contains e, show that there is a tree with smaller cost than T that does not contain e.

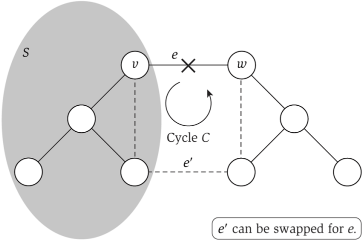

Figure 4.11 Swapping the edge e' for the edge e in the spanning tree T, as described in the proof of (4.20).

Optimality of the Reverse-Delete Algorithm

- Reverse-Delete algorithm: Maintain a set E' of edges.
  - Start with $E' = E$.
  - Process edges in decreasing order of cost.
  - Delete the next edge e from E' only if (V, E') is connected after deletion.
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- Claim: the Reverse-Delete algorithm outputs an MST.
  1. Show that every edge deleted belongs to no MST.
  2. Prove that the graph remaining at the end is a spanning tree.

Comments on MST Algorithms

- To handle multiple edges with the same length, perturb each length by a random infinitesimal amount. Read the textbook.
- *Any* algorithm that constructs a spanning tree by including edges that satisfy the cut property and deleting edges that satisfy the cycle property will yield an MST!

Implementing Prim's Algorithm

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  - Stop when $S = V$.
- Sorting edges takes $O(m \log n)$ time.
- Implementation is very similar to Dijkstra's algorithm.
- Maintain S and store attachment costs $a(v) = \min_{e \in \text{cut}(S)} c_e$ for every node $v \in V - S$ in a priority queue.
- At each step, extract minimum v from priority queue and update the attachment costs of the neighbours of v.
- Total of $n - 1$ $\textsc{ExtractMin}$ and m $\textsc{ChangeKey}$ operations, yielding a running time of $O(m \log n)$.

Implementing Kruskal's Algorithm

- Start with an empty set T of edges.
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- Sorting edges takes $O(m \log n)$ time.
- Key question: "Does adding $e = (u, v)$ to T create a cycle?"
  - Maintain set of connected components of T.
  - $\textsc{Find}(u)$: return the name of the connected component of T that u belongs to.
  - $\textsc{Union}(A, B)$: merge connected components A and B.
- Answering the question: Adding e creates a cycle if and only if $\textsc{Find}(u) = \textsc{Find}(v)$. If not, execute $\textsc{Union}(\textsc{Find}(u), \textsc{Find}(v))$.

Analysing Kruskal's Algorithm

- How many FIND invocations does Kruskal's algorithm need?

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- We will show two implementations of UNION-FIND:
  - Each FIND takes $O(1)$ time, k invocations of UNION take $O(k \log k)$ time in total.
  - Each FIND takes $O(\log n)$ time and each invocation of UNION takes $O(1)$ time.

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- Total running time of Kruskal's algorithm is $O(m \log n)$.

Union-Find Data Structure

- Abstraction of the data structure needed by Kruskal's algorithm.
- Maintain disjoint subsets of elements from a universe U of n elements.
- Each subset has an name. We will set a set's name to be the identity of some element in it.
- Support three operations:
  1. MAKEUNIONFIND(U): initialise the data structure with elements in U.
  2. FIND(u): return the identity of the subset that contains u.
  3. UNION(A, B): merge the sets named A and B into one set.

Union-Find Data Structure: Implementation 1

- Store all the elements of U in an array Component.
  - Assume identities of elements are integers from 1 to n.
  - Component$[s]$ is the name of the set containing s.
- Implementing the operations:

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- Implementing the operations:
  1. $\textsc{MakeUnionFind}(U)$: For each $s \in U$, set $\textsc{Component}[s] = s$ in $O(n)$ time.
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  3. UNION(A, B): merge B into A by scanning COMPONENT and updating each index whose value is B to the value A. Takes $O(n)$ time.
- UNION is very slow because we cannot efficiently find the elements that belong to a set.

Union-Find Data Structure: Implementation 2

- Optimisation 1: Use an array Elements
  - Indices of Elements range from 1 to n.
  - Elements$[s]$ stores the elements in the subset named s in a list.
- Execute Union(A, B) by merging B into A in two steps:
  1. Updating Component for elements of B in $O(|B|)$ time.
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- FIND is fast in the worst case, UNION is fast in an amortised sense. Can we make both operations worst-case efficient?

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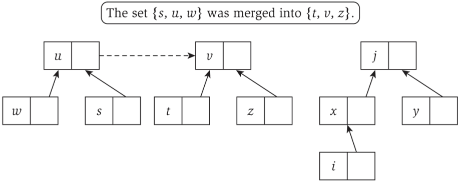

Figure 4.12 A `Union-Find` data structure using pointers. The data structure has only two sets at the moment, named after nodes v and j. The dashed arrow from u to v is the result of the last `Union` operation. To answer a `Find` query, we follow the arrows until we get to a node that has no outgoing arrow. For example, answering the query `Find`(i) would involve following the arrows i to x, and then x to j.

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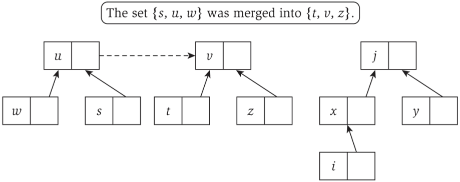

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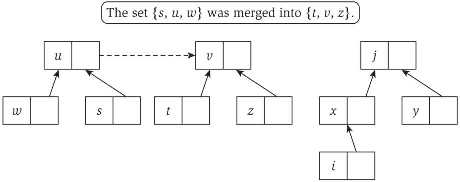
- Implementing UNION(A, B): make smaller tree's root a child of the larger tree's root. Takes $O(1)$ time.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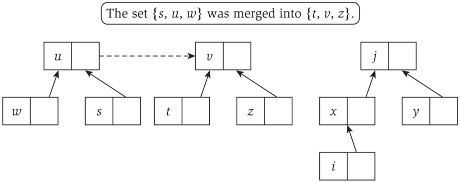
Union-Find Data Structure: Find in Implementation 3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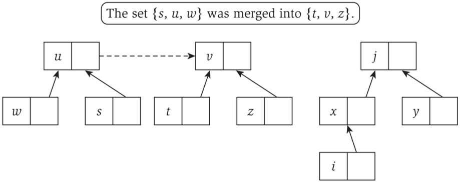
- Why does FIND(u) take $O(\log n)$ time?

Union-Find Data Structure: Find in Implementation 3



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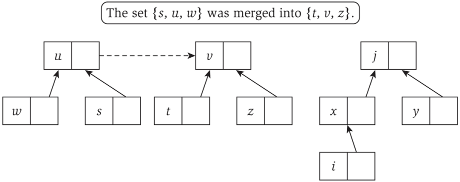
- Why does $\textsc{Find}(u)$ take $O(\log n)$ time?
- Number of pointers followed equals the number of times the identity of the set containing u changed.
- Every time u's set's identity changes, the set at least doubles in size $\Rightarrow$ there are $O(\log n)$ pointers followed.

Union-Find Data Structure: Improving Implementation 3



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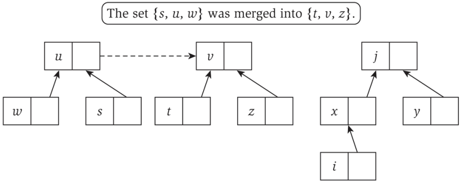
- Every time we invoke FIND(u), we follow the same set of pointers.

Union-Find Data Structure: Improving Implementation 3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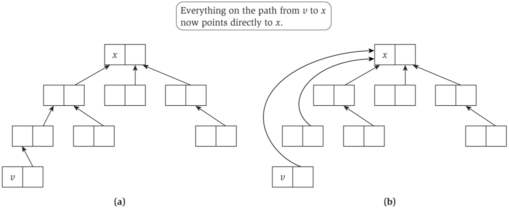
- Every time we invoke $\textsc{Find}(u)$, we follow the same set of pointers.
- Path compression: make all nodes visited by $\textsc{Find}(u)$ children of the root.

Union-Find Data Structure: Improving Implementation 3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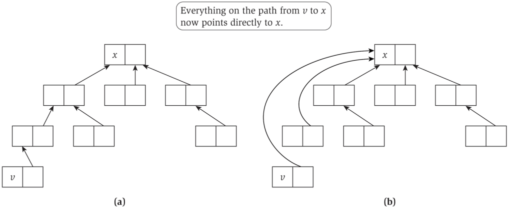
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x
v
(a)
x
v
(b)

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- Every time we invoke Find(u), we follow the same set of pointers.
- Path compression: make all nodes visited by Find(u) children of the root.
- Can prove that total time taken by n Find operations is $O(n\alpha(n))$, where $\alpha(n)$ is the inverse of the Ackermann function, and grows e-x-t-r-e-m-e-l-y s-l-o-w-l-y with n.

Comments on Union-Find and MST

- The UNION-FIND data structure is useful to maintain the connected components of a graph as edges are added to the graph.
- The data structure does not support edge deletion efficiently.
- Current best algorithm for MST runs in $O(m\alpha(m, n))$ time (Chazelle 2000) and $O(m)$ randomised time (Karger, Klein, and Tarjan, 1995).
- Holy grail: $O(m)$ deterministic algorithm for MST.