CS 4104
Review of Mathematical Induction
January 25, 2005

1 Principle of Mathematical Induction

Let $\mathbf{P}$ be some property of the natural numbers $\mathbf{N}$, the set of non-negative integers. Alternately, $\mathbf{P}(n)$ is a statement about a natural number $n \in \mathbf{N}$ that is either true or false. The purpose of induction is to show that $\mathbf{P}(n)$ is true for all $n \in \mathbf{N}$.

Here are three variants of the Principle of Mathematical Induction.

Principle of Mathematical Induction (First Variant). Suppose that we can prove these two statements:

- **Base case.** $\mathbf{P}(0)$ is true.
- **Inductive step.** If $\mathbf{P}(k)$ is true for any $k \in \mathbf{N}$, then $\mathbf{P}(k+1)$ is also true.

Then, by the Principle of Mathematical Induction, $\mathbf{P}(n)$ is true for all $n \in \mathbf{N}$.

Principle of Mathematical Induction (Second Variant). Suppose that $b \in \mathbf{N}$ and that we can prove these two statements:

- **Base case.** $\mathbf{P}(k)$ is true for $0 \leq k \leq b$.
- **Inductive step.** If $\mathbf{P}(k)$ is true for some $k \geq b$, then $\mathbf{P}(k+1)$ is also true.

Then, by the Principle of Mathematical Induction, $\mathbf{P}(n)$ is true for all $n \in \mathbf{N}$.

Principle of Mathematical Induction (Third Variant; Strong Induction). Suppose that $b \in \mathbf{N}$ and that we can prove these two statements:

- **Base case.** $\mathbf{P}(k)$ is true for $0 \leq k \leq b$.
- **Inductive step.** If $k \geq b$ and $\mathbf{P}(i)$ is true for all $i \leq k$, then $\mathbf{P}(k+1)$ is also true.

Then, by the Principle of Mathematical Induction, $\mathbf{P}(n)$ is true for all $n \in \mathbf{N}$.

2 Proof by Induction

An **inductive argument** to prove that a property $\mathbf{P}$ of $\mathbf{N}$ is true for all natural numbers is structured as follows:

Basis. Prove $\mathbf{P}(0)$.

Inductive hypothesis. Assume that $\mathbf{P}(k)$ is true for an arbitrary $k \in \mathbf{N}$.

Inductive step. Prove that the inductive hypothesis implies $\mathbf{P}(k+1)$.

By the Principle of Mathematical Induction (First Variant), $\mathbf{P}(n)$ is true for all $n \in \mathbf{N}$.

3 Example of An Inductive Argument

Prove by induction on n that $n^4 - 4n^2$ is divisible by 3, for all $n \geq 0$.

Base case: If $n = 0$, then $n^4 - 4n^2 = 0$, which is divisible by 3.

Inductive hypothesis: For some $n \geq 0$, $n^4 - 4n^2$ is divisible by 3.

Inductive step: Assume the inductive hypothesis is true for n. We need to show that $(n+1)^4 - 4(n+1)^2$ is divisible by 3. By the inductive hypothesis, we know that $n^4 - 4n^2$ is divisible by 3. Hence $(n+1)^4 - 4(n+1)^2$ is divisible by 3 if $(n+1)^4 - 4(n+1)^2 - (n^4 - 4n^2)$ is divisible by 3. Now

$$\begin{aligned}
(n+1)^4 - 4(n+1)^2 - (n^4 - 4n^2) &= n^4 + 4n^3 + 6n^2 + 4n + 1 - 4n^2 - 8n - 4 - n^4 + 4n^2 \\
&= 4n^3 + 6n^2 - 4n - 3,
\end{aligned}$$

which is divisible by 3 if $4n^3 - 4n$ is. Since $4n^3 - 4n = 4n(n+1)(n-1)$, we see that $4n^3 - 4n$ is always divisible by 3. Going backwards, we conclude that $(n+1)^4 - 4(n+1)^2$ is divisible by 3, and that the inductive hypothesis holds for $n+1$.

By the Principle of Mathematical Induction, $n^4 - 4n^2$ is divisible by 3, for all $n \in \mathbf{N}$.

4 Another Example

Define a set Y with a recursive definition.

A. **Basis:** $7 \in Y$.

B. **Recursive step:** If $y \in Y$, then $y + 21 \in Y$ and $y + 49 \in Y$.

C. **Closure:** The only elements of Y are those obtained from the basis and those obtained from the basis by a finite number of applications of the recursive step.

Prove by induction that every element of Y is divisible by 7.

Base case: The base case of the recursive definition is $7 \in Y$ and 7 is divisible by 7. Hence the statement is true for the base case.

Inductive hypothesis: For some $k \in \mathbf{N}$, every element of Y obtained by k applications of the recursive step is divisible by 7.

Inductive step: Assume that $k \in \mathbf{N}$ and the inductive hypothesis holds for k. Let $y \in Y$ be obtained by $k + 1$ applications of the recursive step. Then, there exists $y' \in Y$ such that y' is obtained by k applications of the recursive step and y is obtained from y' by one application of the recursive step. By the inductive hypothesis, y' is divisible by 7. Either $y = y' + 21$ or $y = y' + 49$; in either case, y is divisible by 7, since y', 21, and 49 are divisible by 7. Hence, every element of Y obtained by $k + 1$ applications of the recursive step is divisible by 7.

By the Principle of Mathematical Induction, every element of Y obtained by a finite number of applications of the recursive step is divisible by 7; hence, all elements of Y are divisible by 7.

5 Exercise in Proof by Induction

Here are two definitions of the set of undirected trees.

First Definition of an undirected tree. An **undirected tree** is an undirected graph that is connected and that contains no cycle.

Second Definition of an undirected tree. The set Z of **undirected trees** is defined recursively by

A. **Basis:** The basis set Z_0 consists of every undirected graph having a single vertex and no edges.

B. **Recursive step:** If T is a tree, then the addition of a new vertex v and an edge from v to any vertex of T results in a tree.

C. **Closure:** The only elements of Z are those in Z_0 and those obtained from Z_0 by a finite number of applications of the recursive step.

Exercise: Show that the two definitions are equivalent (define the same set of graphs).