Appendix for Project 2: deriving the velocity formula

Suppose an object with mass $m$ is dropped from a certain initial altitude and allowed to fall freely; i.e., the only forces acting on it after its release are gravity and the resistance of the air through which it falls:

![Diagram showing forces: $F_D$: drag, $F_G$: gravity]

First, we need a little physics. The force exerted by gravity equals the mass of the object times the gravitational acceleration constant $g$:

$$F_G = m \times g$$

Assuming constant air density (reasonable if the distance fallen is not too great) and a stable orientation of the object as it falls, the force of air resistance on the object is proportional to the velocity of the object squared:

$$F_D = -k \times v^2$$

where $k$ is a constant that depends on the shape of the object and the density of the air through which it falls and $v$ is the velocity of the object. The force is negative since it acts in the opposite direction to the object’s motion.

Now, Newton’s Second Law of Motion says that the total force acting on an object equals the mass of the object times its acceleration $a$, so we have:

$$F = F_G + F_D \quad \text{or} \quad m \times a = m \times g - k \times v^2$$

Now we need a little calculus. The acceleration and velocity are both functions of time $t$, and the acceleration $a$ is the derivative of the velocity $v$. So we have:

$$m \times \frac{dv}{dt} = m \times g - k \times v^2 \quad \text{and also} \quad v(0) = 0$$

since the object isn’t moving when it is released (at time 0). This is an example of a separable nonlinear differential equation with initial condition (covered in Math 1206 and 2214). It is possible to solve for a function $v$ that satisfies both the differential equation and the initial condition.

To start, express in differential notation and do some algebra to separate the variables:

$$\frac{mdv}{mg-kv^2} = dt$$
Now antidifferentiate each side of the equation. This requires a $u-du$ substitution on the left hand side:

$$\int \frac{mdv}{mg - kv^2} = \int dt$$

Let

$$u = \frac{k}{mg} \times v \quad \text{then} \quad du = \frac{k}{mg} dv$$

This substitution transforms the left-hand integral above in the following way:

$$\int \frac{mdv}{mg - kv^2} = \frac{1}{g} \int \frac{du}{1 - \frac{u^2}{\frac{m}{kg}}} = \frac{m}{kg} \tanh^{-1}\left(\frac{\frac{mg}{k} - v}{\frac{mg}{k}}\right) + C$$

The right-hand integral is trivial, and we obtain:

$$\frac{m}{kg} \tanh^{-1}\left(\frac{\frac{mg}{k} - v}{\frac{mg}{k}}\right) + C = t$$

Since the object is dropped from a stationary position at time $t = 0$, we have $v = 0$ when $t = 0$. Substituting that into the equation above yields that $C = 0$. Then, performing algebra to solve for velocity $v$ as a function of time $t$ yields:

$$v = \frac{mg}{k} \tanh\left(\frac{gk}{m} t\right)$$